Calculus of residues.

Thursday, November 2, 2023 12:00 PM

Theorem. $R = \mathrm{Res}_{z=z_0} f(z)$ is the unique number such that

$f(z) - \frac{R}{z-z_0}$ has antiderivative in $B(z_0, dist(z_0, \partial\Omega)) \setminus \{z_0\}$.

Proof. (Uniqueness)

$f(z) - \frac{R}{z-z_0}$, $f(z) - \frac{R'}{z-z_0}$ has antiderivative $\Rightarrow$ $f(z) - \frac{R}{z-z_0} - \left(f(z) - \frac{R'}{z-z_0}\right) = \frac{R'-R}{z-z_0}$ has antiderivative $\Rightarrow$

$\Rightarrow 0 = \oint_{C_r} \frac{R-R'}{\cancel{z-z_0}} dz = 2\pi i(R-R') \Rightarrow R = R'$.

(Existence).

$\gamma \subset B(z_0, dist(z_0, \partial\Omega)) \setminus \{z_0\}$ - closed curve

Then $\gamma \sim 0$ in $B(z_0, dist(z_0, \partial\Omega))$

$C_r \sim 0$

Let $\gamma' := \gamma - n(\gamma, z_0) C_r$. $R = \mathrm{Res}_{z=z_0} f(z)$

Then $n(\gamma', z_0) = n(\gamma, z_0) - n(\gamma, z_0) = 0$, so $\gamma' \sim 0$ in $B(z_0, dist(z_0, \partial\Omega)) \setminus \{z_0\}$.

So $0 = \oint_{\gamma'} f(z)\, dz = \oint_\gamma f(z)\, dz - n(\gamma, z_0) \oint_{C_r} f(z) dz =$

$\oint_\gamma f(z) dz - \underbrace{n(\gamma, z_0)}\, R \cdot \underbrace{2\pi i} = \oint_\gamma f(z)\, dz - \underbrace{\oint_\gamma \frac{R\, dz}{z-z_0}} = \oint_\gamma \left(f(z) - \frac{R}{z-z_0}\right) dz$.

$n(\gamma, z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{dz}{z-z_0}$

$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ $\qquad (z-z_0)^n = \left(\frac{(z-z_0)^{n+1}}{n+1}\right)'$, $n \neq -1$

$R = a_{-1}$

Theorem (Residue Theorem).

Let Ω be a region, $I \subset \Omega$ - a discrete set (i.e. $\forall z \in \Omega\ \exists \delta > 0 : (B(z,\delta) \setminus \{z\}) \cap I = \emptyset$).

Let $\gamma \subset \Omega$ - a cycle, $\gamma \cap I = \emptyset$, $\gamma \sim 0$ in Ω.

Let $f \in A(\Omega \setminus I)$.

Then: $\frac{1}{2\pi i} \oint_\gamma f(z)\, dz = \sum_{w \in I} n(\gamma, w)\, \mathrm{Res}_{z=w} f(z)$

Remark. The set $I_{n_\gamma} = \{w : n(\gamma, w) \neq 0\}$ is bounded and $\subset \Omega$. Thus $I \cap I_{n_\gamma}$ is finite (otherwise, $\exists z_j \in I \cap I_{n_\gamma}$, $z_j \to z \in \Omega$. z does not satisfy the discreteness).

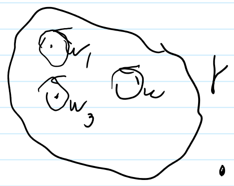
So the sum on RHS is finite! ∎

Proof. Let $w_1, \dots, w_k$ be singularities in I_{n_γ}. $I_{n_\gamma} \cap I = \{w_1, \dots, w_k\}$.

Proof. Let $w_1, \dots, w_K$ be singularities in $\mathrm{In}\,\gamma$. $\mathrm{In}_\gamma \cap I = \{w_1, \dots, w_K\}$.

Let us choose $r_k > 0$: 1) $r_k < \mathrm{dist}(w_k, \gamma)$

2) $r_k < \dfrac{|w_j - w_k|}{4}$ $\forall j \neq k$.



Let $C_k := \{w_k + r_k e^{it}\}$.

Then, as before: $\gamma' := \gamma - \sum n(\gamma, w_k) C_k \sim 0$ in $\Omega \setminus I$.

(because $n(\gamma - \sum n(\gamma, w_k) C_k, w_j) = n(\gamma - n(\gamma, w_j) C_j, w_j) =$

$n(\gamma, w_j) - n(\gamma, w_j) = 0$).

$= n(\gamma, w_j) - \sum \underbrace{n(\gamma, w_k)}\, n(C_k, w_j)$ $\quad n(C_k, w_j) = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$

So $\oint_{\gamma'} f(z)\,dz = 0 \Rightarrow$ $\displaystyle\oint_\gamma f(z)\,dz = \sum_{k=1}^{n} n(\gamma, w_k) \underbrace{\oint_{C_k} f(z)\,dz}_{2\pi i \operatorname{Res}_{z=w_k} f(z)}$

Corollary. Let γ be the oriented boundary of a region Ω. $I = \{z_1, \dots, z_n\} \subset \Omega$ – finite set, $f \in A((\Omega \cup \gamma) \setminus I)$. Then

$$\frac{1}{2\pi i} \oint_\gamma f(z)\,dz = \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f(z).$$